Quanta of the Third Kind

Frank Wilczek

Quantum mechanics is nearly one hundred years old, and yet the challenge it presents to the imagination is so great that scientists are still coming to terms with some of its most basic implications. Here I will describe some theoretical insights and recent experimental results that are leading physicists to revise and expand their ideas about what quantum-mechanical particles are and how they behave. These new ideas are centered around a topic traditionally known as quantum statistics. The name is misleading: the basic physical phenomena do not involve statistics in the usual sense. A better title might have been the quantum mechanics of identity, but the new developments make that name obsolete too. A more accurate description would be the quantum mechanics of world-line topology. Since that is quite a mouthful, most researchers now simply refer to anyon physics.

Quantum mechanics achieves a strange and wonderful unification between forms of matter that appear to be vastly different. Prior to the advent of quantum theory, electrons and atomic nuclei were regarded as particles, conforming to Isaac Newton's classic definition: "hard, massy, impenetrable." During the nineteenth century, light came to be understood in terms of waves or, ultimately, space-filling electromagnetic fields. That description supplies a rich and accurate account of interference, diffraction, and many aspects of the interaction between light and matter.

In quantum theory, electrons, light, and all other forms of matter are described using the same mathematics. A more general concept, sometimes expressed as the *wavicle*, governs everything. The wavicle is a space-filling function—the wave function—that describes the probability of finding a particle at different places. This common framework accommodates both the wave behavior of electrons, manifested in electron diffraction, and the particulate nature of light, manifested in the all-or-none response of the photoreceptor cells responsible for color vision.

Most popular and even introductory textbook accounts of quantum theory stop there. But the great wavicle unification has an important qualification: it applies only to single particles. When we compare the quantum description of two or more electrons with that of two or more photons, we find fundamental differences.

Electrons, neutrons, and protons are examples of fermions, named in honor of Enrico Fermi, who pioneered their study.[1] Fermions are antisocial by nature. More precisely, they obey the Pauli exclusion principle,[2] which states that no two fermions of the same kind can be in the same quantum state. The exclusion principle plays a central role in our understanding of atoms, atomic nuclei, white dwarfs, neutron stars, and matter in general:

- In many-electron atoms, the exclusion principle forces the electrons to occupy different orbitals. This behavior is essential for building up the shell structure of atoms, which underlies the periodic table of elements and chemistry.
- Similarly, in atomic nuclei, the exclusion principle governs the behavior of the protons and neutrons, building up the shell structure that controls the nuclear chemistry of fission and fusion.
- Freeman Dyson and Andrew Lenard demonstrated mathematically that if the equations of quantum theory are applied to the ingredients of ordinary matter without taking into account that electrons are fermions, the mixture implodes.[3] Once the electrons are treated as fermions, all is well.
- White dwarfs are the evolutionary final state of moderate-sized stars, such as our sun. After they have exhausted their nuclear fuel, these stars collapse to much smaller sizes. The sun, for example, will eventually become Earth sized. Beyond a certain point, the electrons within these objects prevent further compression because too small an object would contain too few orbitals to accommodate them.
- Neutron stars are the final stage in the evolution of somewhat larger stars. Subrahmanyan Chandrasekhar demonstrated that there exists a limit to how much pressure electrons can withstand.[4] Past the so-called Chandrasekhar limit, stellar remnants

> up to about twice the sun's mass collapse further, down to a few kilometers in radius, whereupon the exclusion principle for neutrons halts the process.

Due to their Fermi statistics, identical fermions exhibit a repulsive force of quantum-mechanical origin that is over and above the four conventional forces of the Standard Model—strong, weak, electromagnetic, and gravitational. That effective force is not merely an esoteric addition to the basic forces, but a central pillar in our understanding of nature.

Photons, together with gravitons, the Higgs particle, and many other particles, are examples of bosons,[5] named in honor of Satyendra Bose, who was the first to study them.[6] In contrast to fermions, bosons are natural conformists and prefer to be in the same state. The probability for multiple occupancy, which vanishes for fermions, is enhanced for bosons, which are said to obey Bose statistics.

Laser beams epitomize the bosonic behavior of photons. Within a laser beam, many photons have succeeded in occupying the same state with the same color, same direction, and same spatial profile. More complex material manifestations of Bose statistics are superfluidity and superconductivity. In those low-temperature states of matter, large numbers of 4He atoms or Cooper pairs of electrons, respectively, occupy the same quantum state and thus flow coherently. At low temperatures, when they are less distracted by the noise of the external world, one might say that bosonic particles get to do what they want to do—which is to do the same thing.

Quantum mechanics coalesced during the 1920s. Decades of adventurous discovery followed, in which many new particles were identified. The behavior of these particles involved many qualitatively new phenomena, including antimatter, strangeness, oscillatory changes in identity, and violation of spatial parity and time-reversal symmetry. Particles that could not be observed in isolation—quarks and gluons, which are fermions and bosons,[7] respectively—became fundamental ingredients in our best description of nature. During all these upheavals, the division of the world of particles into just two kingdoms, those of fermions and bosons, remained intact. By the 1970s these notions had become conventional wisdom, bordering on dogma.

In 1977, two Norwegian physicists, Jon Leinaas and Jan Myrheim, challenged that consensus.[8] Subsequent investigation has clarified the profound roots of quantum statistics, why bosons and fermions are so pervasive, and the possibility of alternatives. Before proceeding further, it is worth taking a moment to review the ultimate source of boson and fermion behavior, as presently understood.

The fundamental task of quantum mechanics is to calculate the probability for a specified event to occur. This is done by calculating an auxiliary quantity, the *amplitude* of the event, and then squaring the amplitude to obtain the probability.[9]

There are several ways to calculate quantum mechanical amplitudes. The most transparent method was discovered by Richard Feynman.[10] The "sum over histories" approach involves a consideration of all possible ways in which a process might have happened. The dynamical description of the system provides a definite mathematical rule, or algorithm, that assigns a numerical base-amplitude to each possible history. The total amplitude is obtained by adding all the base-amplitudes. In this framework, the central task of fundamental quantum theory is to discover the rules for calculating base-amplitudes in different physical situations. Physicists often look to classical physical for guidance, because for large objects the quantum rules must reproduce observed classical behavior.

With that framework in mind, consider a process in which two indistinguishable particles—two electrons, say, or two photons—start at two positions (A, B) and end up at two other positions (C, D). The possible histories underlying this process fall into two distinct classes. In one class, the particle originating at A travels to C, while the particle originating at B travels to D. In the other class, the particle at A travels to D while the particle at B travels to C. Since the particles are indistinguishable, one cannot tell, by looking only at the outcome, which class of historical process led to it. Guided by classical physics, physicists can develop rules for how to assign base-amplitudes within each of the two classes. Adding the base-amplitudes within each class yields two partial amplitudes.

The remaining issue is to determine a rule for combining the two partial amplitudes into the total amplitude. Classical physics offers no guidance here. Indeed, classical physical theory assures us that in principle we can keep accurate tabs on particles. But if that is the case, the two topologically distinct classes of histories correspond to physically distinct processes, each of which is characterized by a separate probability. In quantum theory, on the contrary, one cannot keep tabs. That is an aspect of Heisenberg's uncertainty principle, which limits how well a particle's position can be resolved. When the uncertainties in the positions of our two particles overlap, it becomes impossible to keep track of who is who.[11]

The rule for combining the two partial amplitudes must then involve some essentially new consideration that goes beyond classical physics. Quantum statistics, with all its weighty implications for physics, ultimately comes down to this rule. The traditional rules are as follows: for bosons, add them, and for fermions, subtract them. These are the only two available choices. These are the only two available choices, it seems, because quantum theory imposes an important general consistency requirement.[12] If we apply our rule twice to the process (A, B) → [(A, B) or (B, A)] → (A, B)—we must obtain the same result as we get by applying it directly—e.g., to (A, B) → (A, B). Thus, since a double exchange is equivalent to no exchange at all, the factor x that we can associate with an exchange must satisfy $x^2 = 1$. This implies that either $x = 1$, as for bosons, or $x = -1$, as for fermions.

This elegant and superficially profound understanding of why there can be bosons and fermions, and nothing else, relies on an important implicit assumption that escaped attention of physicists for more than fifty years. Consider two particles whose motion is confined to two dimensions—specifically, a plane. To carry out the sum over histories, one must consider how the particles move in time as they progress from (A, B) to (C, D). In visualizing this problem, it is convenient to regard time as a third dimension, perpendicular to the two spatial dimensions. The motion of each particle then defines a path in a three-dimensional space-time known as a *world-line*.

The world-lines of two particles can wind around one another; and if they do, there is a discrete topological distinction among the histories from (A, B) to (C, D)—namely, the number of times the world-line of the first particle winds around the world-line of the second particle. Mathematicians refer to this as the winding number.[13] In cases of more than two particles, world-lines can become interwoven in elaborate patterns termed *braids*.

For particles that move in three dimensions, the need to consider winding and braiding processes no longer arises. The contrast between the rich topology of multiparticle histories in two dimensional spaces—i.e., three-dimensional space-times—and the paltry topology of multiparticle histories in three-dimensional spaces—i.e., four-dimensional space-times—is closely related to the fact that in four dimensions, though not of course in three, knots are always easy to untangle.[14] In the topology of braids, less is more, since a smaller ambient space means less room for untangling maneuvers.

In three dimensions, the only two topologically distinct classes of histories are the ones involving exchanges of position. Given that mathematical fact, the potentially profound argument given in the previous section then becomes definitive. The meagre topology of multiparticle paths in three-dimensional space offers only the choice between bosons and fermions, with no other options. This is a very satisfying result because it justifies the classification into bosons and fermions that physicists had discovered empirically.

By contrast, the richer topology of multiparticle paths in two-dimensional space supports a much bigger menu of consistent quantum-mechanical rules. One can add partial amplitudes that arise from infinitely many topologically distinct classes, and the consistency conditions are less constraining. In a 1982 paper, I introduced the term *anyon* to describe this situation, with the connotation that anything goes.[15] While it is not literally true that anything goes, theoretically speaking, the move to flatland opens up many new possibilities for the quantum statistics of particles. Indeed, the kingdom of anyons has many mansions.

The consistent rules for quantum statistics in two space dimensions can become quite complicated—at least as complicated as braids. I will describe here only the very simplest anyons precisely, and briefly acknowledge some others. Even in the simplest case, the anyon rule for combining different partial amplitudes uses basic concepts about complex numbers.[16] There are different topological classes of braids, distinguished among other things by the number of times different strands wind around one another. A rule is needed for combining the partial amplitudes from those sectors. The simplest anyon rule is as follows: multiply the partial amplitude for each class by the complex number $e^{i\alpha W}$, where W is equal to the total number of windings, counting all pairs of particles. Different values of α define different species of anyons.

More structured, so-called non-abelian anyon rules are sensitive to other details associated with braids. These rules usher in more complicated wave functions that are not simply complex numbers, but arrays of numbers. The different components in the array represent possible values of an emergent degree of freedom, roughly analogous to the possible colors of quarks. When non-abelian anyons wind around one another, their joint wave function is transformed by more complicated operations than multiplication by a number—that is, multiplication by a unitary matrix. In this way, non-abelian anyons acquire a strange, capacious storage capacity. Their quantum-mechanical wave functions carry a more detailed record of the braids their world-lines build up, which tracks more information than total winding.

The possible rules defining different species of non-abelian anyons are intricate and diverse. Here, a few names and references will have to suffice: Ising anyons, Fibonacci anyons, parafermions of several kinds, and Majorinos.[17] It is fascinating to observe that world-lines can wind even among particles that are not indistinguishable. That possibility, which I named *mutual statistics*, highlights the novelty of the new, more general perspective on quantum statistics.

These extraordinary new possibilities for physical behavior are fun to think about, but they might also seem somewhat academic or fantastical. After all, we do not live in flatland. But we can still visit. In fact, the physical world abounds in flatlands, and they play starring roles in modern technology. Planar circuits photolithographically etched onto layered surfaces, otherwise known as chips, are the workhorses of modern microelectronics. In electronic chips, the motion of electrons is essentially confined to two dimensions. If the electrons that live on a chip were zapped with enough energy they could be removed. But so long as their energies don't get too big, the electrons are confined to two dimensions, and the quantum mechanics of flatland applies.

Physicists have learned to organize their fundamental descriptions of the quantum world by focusing on the behavior of energy concentrations that are reasonably stable and exhibit reproducible properties. Such entities are termed elementary particles, and they are used as the building-blocks in our best model of the physical world.

In thinking about exotic materials and states of matter, it has been fruitful to consider them as worlds in themselves: *quasi-worlds*, inhabited by *quasiparticles*.

Suppose that a crystalline solid is zapped with a well-focused laser pulse and an electron is ejected. The remaining material will then contain a localized unit of positive charge where the electron used to be. After radiating some excess energy, this excitation may settle into a stable, reproducible form—a quasiparticle. This kind of quasiparticle, which reflects the absence of an electron, is usually referred to as a hole. In semiconductors, holes are units of positive charge that are cheap to produce and much easier to move than protons. Understanding the properties of holes was a key step in the invention of solid-state transistors and the emergence of modern microelectronics.

That success story, and others like it, has inspired some physicists to cultivate an art that might be described as designing quasi-worlds.[18] To begin with, one must imagine quasi-worlds with interesting properties, and then seek out or manufacture materials and states of matter that embody them.[19] Of course, that strategy can only work if the quasi-worlds are not too outlandish. Success requires discipline and good taste, as well as inspiration.

I first began to consider the ideas and new possibilities for quantum statistics in 1982. At the start, I was unaware of the work of Leinaas and Myrheim, which had attracted little attention. I was simply imagining quasi-worlds. In my conceptions, three lines of thought came together:

- Fractionalization: properties of particles that ordinarily appear only as whole-number multiples of a fundamental unit might occur in smaller multiples within quasi-worlds. Roman Jackiw and Claudio Rebbi abstractly, and Wu-Pei Su, Robert Schrieffer, and Alan Heeger concretely, demonstrated that quasiparticles could carry half a unit of electric charge, i.e., half the charge of an electron.[20] Jeffrey Goldstone and I had shown that in other quasi-worlds different fractions of charge could occur.[21] I wanted to see if a similar fractionalization could happen for angular momentum, that is, spin.[22]
- Flux tubes: I soon realized that fractional angular momentum was indeed possible by means of a very specific physical mechanism: particles orbiting around tubes of magnetic flux. That was an encouraging result, because the theory of flux tubes was already a well-developed, respectable subject. Flux tubes occur in a large class of superconductors, so-called type II superconductors, and in promising, though speculative, unified field theories.
- Dimensional reduction: from a slice of a narrow tube, one can obtain a small, essentially point-like structure that can be considered a particle. Thus, the calculated behavior of narrow tubes in three-dimensional space could be used to construct new kinds of particles in two-dimensional space.

When I gave a seminar about these ideas at Caltech, my friend and colleague John Preskill reminded me that there is a deep connection between the spin of a particle and its quantum statistics.[23] If there is fractional spin, he asked, shouldn't there also be fractional statistics? This was a question I had not considered. In fact, I didn't see right away what the term fractional statistics could even mean. On the drive back to Santa Barbara, I realized that the right thing to think about was braiding, and that braiding flux tubes would yield behavior that could be interpreted as fractional quantum statistics. Within a few days I pulled my thoughts together in two short papers.[24] Anyons had now acquired a name and a more-or-less plausible, though not yet concrete, path to physical embodiment.

Strange things can happen when one exposes two-dimensional droplets of mobile electrons—in a narrow range of densities and held at ultralow temperatures—to strong magnetic fields. Under these conditions, as one varies the strength of the magnetic field, a large family of new states of matter is produced. These states, known as fractional quantum Hall liquids (FQHL) are interrelated, but still distinctly different. Though the required conditions are very special and hard to achieve, the properties of FQHL states are so new and interesting that they have become a preoccupation for many researchers since FQHLs were first discovered in 1982.

The most direct physical manifestation of FQHL states is the phenomenon referred to as quantized Hall conductance. This can be observed if one puts four leads—A, B, C, and D, in that order—at different points on the boundary of the sample. A voltage is applied across A and C by connecting them through a battery, for example. The current flowing between B and D is then measured. As one varies the magnetic field, the conductance—that is, the ratio of voltage to current—does not always vary continuously. Instead, it holds constant around a series of so-called plateau values.

To the uninitiated this behavior might seem an esoteric curiosity, but to physicists it came as a shocking departure from prior experience and expectations. Each plateau represents a new state of matter—a distinctive quasi-world, with its own quasiparticles. Robert Laughlin shared the 1998 Nobel Prize in Physics for his theoretical elucidation of the phenomenon, together with Horst Störmer and

Daniel Tsui, who discovered the effect experimentally.[25] Laughlin's theory remains the foundation of our present understanding.

For the purposes of the current discussion, the most relevant part of Laughlin's theory, originally published in 1983, is his picture of quasiparticles in the FQHL states. In short, one produces a quasiparticle by subjecting the material to the influence of a notional flux tube.[26] The resulting quasiparticles have remarkable properties, that differ from one plateau to another. Notably, their electric charge is a fraction of the electric charge of an electron. In the FQHL state that is easiest to produce, the one-third state, the electric charge of a quasiparticle is one-third of the charge of an electron, or $e/3$. There is also a one-fifth state on a different plateau, where the quasiparticles have charge $e/5$. There are many other states in which the quasiparticle charges are other fractions.

My colleagues Daniel Arovas and Schrieffer were aware of my interest in the theory of fractional quanta and provided tutorials on the nascent theory of the FQHL. When we came to the part about flux tubes, I was able to teach them something in return: the ideas about anyons, and their realization through flux tubes, that I sketched above. Within a few days, we figured out how to bring those general ideas to bear on the FQHL. In a paper published in 1984, we demonstrated mathematically that the FQHL quasiparticles *are* anyons, in the precise sense that when you move them around each other—that is, when you braid their world-lines—their wave function does exactly what anyon wave functions are supposed to do.[27] At the time, I thought it would be easy to test our prediction experimentally. In the years that followed, many people tried to do just that. But nobody succeeded, due to a variety of technical challenges. It was not until 2020 that two groups were able to obtain decisive results.

Experiment One: Levels of Conformity

Hugo Bartolomei et al. set up a kind of quasi-accelerator within the quasi-world of the 1/3 FQHL state.[28] As part of their experiment, they produced channels in the shape of an X through which quasiparticles could flow and injected quasiparticle beams flowing upward at the bottom. At the crossing, quasiparticles could meet and scatter from one another. By studying the output at the top, one obtains information about how the quasiparticles interact, through a process physicists term *scattering*.

The quantum statistics of the quasiparticles affects how they scatter in ways that can be calculated confidently. Since two bosons like to do the same thing, they will have a much-enhanced probability to scatter in the same direction, i.e., to enter the same upper arm. Fermions, on the contrary, will strongly prefer to enter different arms. The observed results, falling in between, fit neither of those expectations. Instead, they align well with predictions derived from the kind of anyon quantum statistics predicted for the 1/3 FQHL state.[29]

Experiment Two: The Beauty of Braiding

James Nakamura et al. set up an ingenious arrangement modeled on interferometers, a workhorse tool in optics.[30] The central idea in interferometry is to offer a light beam—or quasiparticle—two alternative paths from source to detector. Influences that alter the balance of the paths then show up as changes in the output.

In their experiment, which was likewise performed in the 1/3 FQHL state, a flow of quasiparticles can follow one of two paths that together enclose an island within the sample. From the perspective of a quasiparticle within that island, those two paths differ by a winding.[31] If the quasiparticles are anyons, each island anyon will alter the way the sub-amplitudes for the flowing anyons should be added together, in a predictable way. Thus, anytime an additional anyon appears in the island, the output will suddenly jump, also in a predictable way. This is what they observed.

The beauty of this experiment is how clearly it maps onto the most basic defining characteristic of anyons, namely their response to braiding. The jumps can also be measured accurately, which allows for a clean quantitative comparison with the theoretical predictions. Thankfully, they are in agreement.

Why were Nakamura et al. able to succeed where many others fell short? Over the years there have been steady improvements in the purity of the materials and in the techniques available for setting up tiny currents and measuring them accurately. The crucial innovation added by Nakamura et al. was to surround the sample with a bath of electrons that can move to compensate for inhomogeneities of charge within the sample. The compensation process damps out other forces, while leaving the effect of quantum statistics intact. As a result, the behavior of the quasiparticles becomes more reproducible, and more clearly dominated by their quantum statistics. The strategy employed in this experiment should be adaptable to other FQHL states, which are predicted to support other kinds of anyons.

Experiment Three: Switching from Afar

Using related ideas and as part of work that has extended over several years, Robert Willett et al. applied interferometry to several FQHL states, including some that are suspected, theoretically, to contain non-abelian anyons.[32] In this scenario, the addition of anyons to the central island, one at a time, changes the behavior of the output in different ways at each step. The anyons create something akin to a sophisticated toggle switch.

As part of their work, Willett et al. collected signals that were consistent with theoretical predictions for anyons. At present, their experiments seem to call for more cautious interpretation than those of Nakamura et al., mainly due to questions about the integrity of the island. That said, there is every reason to think that with further work these delicate experiments will become more clear-cut, while their central conclusions will remain valid.

Experiment Four: Engineering Anyons

To conclude this brief survey of recent experimental results, it is appropriate to mention a rather different kind of endeavor that involves realizing anyons within designed systems, as opposed to in natural states of matter.

Collaborators from Austria, China, and Germany set up ingenious circuits involving a mix of conventional and superconducting electronics that support two different kinds of localized excitations.[33] The circuits were designed so that those two kinds of quasiparticles would exhibit nontrivial mutual statistics—and indeed they do.

This construction is meant to be more than a one-off demonstration. It is part of a program to produce fault-tolerant elements for use in quantum computers, based on the ideas of Alexei Kitaev.[34]

Existing realizations of anyons in fractional quantum Hall states are not an ideal vehicle for detailed studies or possible applications. This is because the realizations require ultrapure materials, ultralow temperatures, and high magnetic fields.

It seems possible that different realizations might be free of these drawbacks. Anyons have been predicted to occur in many other quasi-worlds of two-dimensional matter. Numerical simulations have offered support for these ideas. In some cases, there is also suggestive evidence for the predicted behaviors. Crucial experimental tests have been proposed, but they are technically challenging, and they have not yet been carried out.[35]

Another topic for future research is what happens when many anyons are close together, in the same material. Fermion and boson behaviors, as noted previously, have dramatic consequences for many-particle systems. Even simple anyons are predicted to support a rich variety of collective behaviors, including a new mechanism of superconductivity.[36] The behavior of an ensemble of more complex anyons and mixtures containing several kinds of anyons with mutual statistics remains largely unexplored.

Several ancient Andean civilizations, including the Inca, developed a versatile and nonverbal method to record and process information that served them well for many centuries. *Quipus* are formed from a sequence of colored strings containing knots.[37] Each of the strings is tied at one end to a common cord, so that when the cord is suspended the strings hang down and can be scanned easily. The colors of the strings and the placement of the knots might convey an accounting ledger, a historical chronicle, or a military roster. In the quantum world, one might imagine using braids to represent information in a similar manner.[38] Anyons empower such an approach, because the wave functions of multi-anyon systems store memories of the braids formed by their world-lines.

The memory capacity of the simplest anyons, which only keep track of the total winding, is very limited. But well-designed systems that use more complex anyons, whose wave-function rules bring in non-abelian and mutual statistics, can capture much more detailed information. Such systems are the basis for topological quantum computing.[39]

This way of representing information could have important advantages:

- Capacity: the storage capacity of multi-anyon braids grows exponentially fast as the number of anyons or the length of the braids increases, quickly outstripping the capacity of more conventional memories.
- Parallelism: in weaving a single strand through the others, one is confronted with many choices that can produce many different new braids. From the opposite perspective, a single anyon operates in parallel on the information that the preexisting braids encode.
- Noise immunity: the two preceding advantages are characteristic of quantum computers in general. The most distinctive advantage of anyons is their potential reliability. Anyons store and manipulate information about braids, and braids retain their overall form—their topology—even if they are jostled a bit.[40] Since the main technical challenge in quantum computing is avoiding errors, ensuring reliability is a big deal.

Topological quantum computing is currently an active area of research. Microsoft has made substantial investments in the area, and has put forward a concrete, long-range plan for making it into a practical, largescale technology. The process will be far from easy, but the challenges appear more technical than fundamental. In this, as on several other fronts mentioned above, anyons will keep physicists fruitfully engaged for years to come.

Frank Wilczek is a Nobel Laureate, Herman Feshbach Professor of Physics at MIT, founding director of the T. D. Lee Institute, and chief scientist at the Wilczek Quantum Center in Shanghai.

1. Enrico Fermi, "Sulla quantizzazione del gas perfetto monoatomico," *Rendiconti Lincei* 3 (1926): 145–49; see also, Paul Dirac, "On the Theory of Quantum Mechanics," *Proceedings*

of the Royal Society of London, Series A 112, no. 762 (1926): 661–77, doi:10.1098/rspa.1926.0133.

2. Wolfgang Pauli, "Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren," *Zeitschrift für Physik* 31, no. 1 (1925): 765–83, doi:10.1007/bf02980631.
3. Freeman Dyson and Andrew Lenard, "Stability of Matter. I," *Journal of Mathematical Physics* 8, no. 3 (1967): 423–34, doi:10.1063/1.1705209.
4. Subrahmanyan Chandrasekhar, "XLVIII. The Density of White Dwarf Stars," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 11, no. 70 (1931): 592–96, doi:10.1080/14786443109461710.
5. In general, assemblies containing an even number of fermions behave as bosons, while assemblies containing an odd number of fermions behave as fermions. For example, the most common kind of helium atom, ^{4}He, contains a nucleus of two protons and two neutrons, surround by two electrons. Thus, it has six fermions in all, and so it is a boson. ^{3}He atoms contain one neutron fewer, so they are fermions. Reflecting this difference, purified liquids based on ^{4}He and ^{3}He behave drastically differently at low temperatures. Their contrasting behavior is widely exploited in modern cryogenics.
6. Satyendra Bose, "Plancks Gesetz und Lichtquantenhypothese," *Zeitschrift für Physik* 26, no. 1 (1924): 178–81, doi:10.1007/bf01327326; see also Albert Einstein, "Quantentheorie des einatomigen idealen Gases," *Sitzungsberichte der Preußischen Akademie der Wissenschaften* 1 (1925): 3–14.
7. The so-called color of quarks was originally introduced as a distinction without a difference, to allow quarks to be described as fermions. Years later, within quantum chromodynamics, the color degree of freedom got promoted into a new kind of charge, generalizing electric charge. Indeed, this is the crux of quantum chromodynamics.
8. Jon Leinaas and Jan Myrheim, "On the Theory of Identical Particles," *Il Nuovo Cimento B* 37, no. 1 (1977): 1–23, doi:10.1007/bf02727953; see also Frank Wilczek, "Magnetic Flux, Angular Momentum, and Statistics," *Physical Review Letters* 48, no. 17 (1982): 1,144–46, doi:10.1103/physrevlett.48.1144; Gerald Goldin, Ralph Menikoff, and David Sharp, "Representations of a Local Current Algebra in Nonsimply Connected Space and the Aharonov–Bohm Effect," *Journal of Mathematical Physics* 22, no. 8 (1981): 1,664–68, doi:10.1063/1.525110; and Yong-Shi Wu, "General Theory for Quantum Statistics in Two Dimensions," *Physical Review Letters* 52, no. 24 (1984): 2,103–106, doi:10.1103/physrevlett.52.2103.
9. In general, the amplitude is a complex number, and the probability is its absolute square.
10. Richard Feynman and Albert Hibbs, *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill, 1965).
11. Less intuitively but more precisely, according to quantum theory, if we do not measure the position of a particle then it simply does not have a definite position. In this situation of ignorance, we must allow for all virtual possibilities and add up their base amplitudes.
12. But see below!
13. To do the counting properly we should count the total winding in, say, the counterclockwise direction. Clockwise windings are then assigned negative numbers. With this understanding, the total winding for two successive histories is the numerical sum of the windings for each history separately.
14. We can move different parts of the candidate knot through one other, while avoiding collisions, by maneuvering into the fourth dimension. Specifically, if two strands threaten to collide, we can avoid the collision by nudging the strand we want to move a little in the direction that lies outside the three-dimensional space defined by its desired but obstructed direction of motion and the two tangents to the strands.
15. Wilczek, "Magnetic Flux."
16. I will not make a long detour to explain that bit of mathematics. Nothing else in this article makes use of it, so if it's unfamiliar, you can just skip to the next paragraph without losing the thread.
17. Chetan Nayak et al., "Non-Abelian Anyons and Topological Quantum Computation," *Reviews of Modern Physics* 80, no. 3 (2008): 1,083–59, doi:10.1103/revmodphys.80.1083.
18. In the physics literature, it would be called something like, "constructing effective Lagrangians to predict prospective universality classes."
19. The theorists who do the imagining are usually different people from the experimenters who do the making.
20. Roman Jackiw and Claudio Rebbi, "Solitons with Fermion Number ½," *Physical Review D* 13, no. 12 (1976): 3,398–409, doi:10.1103/physrevd.13.3398; and Wu-Pei Su, J. Robert Schrieffer, and Alan Heeger, "Solitons in Polyacetylene," *Physical Review Letters* 42, no. 25 (1979): 1,698–701, doi:10.1103/physrevlett.42.1698.
21. Jeffrey Goldstone and Frank Wilczek, "Fractional Quantum Numbers on Solitons," *Physical Review Letters* 47, no. 14 (1981): 986–89, doi:10.1103/physrevlett.47.986.
22. The angular momentum of particles ordinarily is a whole-number multiple of a minimal unit that is equal to one half of Planck's constant.
23. According to the spin-statistics theorem, which is one of the profound general consequences of relativistic quantum field theory, particles whose spin is an odd multiple of the basic unit are fermions, while those whose spin is an even multiple of the basic unit are bosons. Of course, that theorem, as stated, only applies to particles that live in three-dimensional space. In two dimensions there still is a tight connection between quantum statistics and spin, but the spin is less constrained, and so is the statistics.
24. Frank Wilczek, "Remarks on Dyons," *Physical Review Letters* 48, no. 17 (1982): 1,146–49, doi:10.1103/physrevlett.48.1146; and Frank Wilczek, "Magnetic Flux."
25. Robert Laughlin, "Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged

Excitations," *Physical Review Letters* 50, no. 18 (1983): 1,395–98, doi:10.1103/physrevlett.50.1395; and Daniel Tsui, Horst Störmer, and Arthur Gossard, "Two-Dimensional Magnetotransport in the Extreme Quantum Limit," *Physical Review Letters* 48, no. 22 (1982): 1,559–62, doi:10.1103/physrevlett.48.1559.

26. In this process, the final magnitude of the flux is a critical value called a flux quantum. At this value, the flux tube becomes invisible to the electrons. Technically, it can be compensated by a so-called large gauge transformation. Nevertheless, a quasiparticle is left behind, as a scar from the process of turning the flux tube on.
27. Daniel Arovas, J. Robert Schrieffer, and Frank Wilczek, "Fractional Statistics and the Quantum Hall Effect," *Physical Review Letters* 53, no. 7 (1984): 722–23, doi:10.1103/physrevlett.53.722. At about the same time, Bert Halperin independently inferred that FQHL quasiparticles are anyons using less direct arguments. Bert Halperin, "Statistics of Quasiparticles and the Hierarchy of Fractional Quantized Hall States," *Physical Review Letters* 52, no. 18 (1984): 1,583–86, doi:10.1103/physrevlett.52.1583.
28. Hugo Bartolomei et al., "Fractional Statistics in Anyon Collisions," *Science* 368, no. 6,487 (2020): 173–77, doi:10.1126/science.aaz5601.
29. Bernd Rosenow, Ivan Levkivskyi, and Bertrand Halperin, "Current Correlations from a Mesoscopic Anyon Collider," *Physical Review Letters* 116 (2016): 156801, doi:10.1103/physrevlett.116.156802.
30. James Nakamura et al., "Direct Observation of Anyonic Braiding Statistics," *Nature Physics* 16 (2020): 931–36, doi:10.1038/s41567-020-1019-1.
31. More precisely, if a quasiparticle takes one path from source to detector, and then returns by the other path, it will have wound once around a quasiparticle within the island.
32. Robert Willett et al., "Magnetic-Field-Tuned Aharonov–Bohm Oscillations and Evidence for Non-Abelian Anyons at $\nu = 5/2$," *Physical Review Letters* 111, no. 18 (2013), doi:10.1103/physrevlett.111.186401.
33. Roughly speaking, they involve concentrations of charge and of circulating current. Chao-Yang Lu et al., "Demonstrating Anyonic Fractional Statistics with a Six-Qubit Quantum Simulator," *Physical Review Letters* 102 (2009): 030502, doi:10.1103/physrevlett.102.030502.
34. Han-Ning Dai et al., "Four-Body Ring-Exchange Interactions and Anyonic Statistics within a Minimal Toric-Code Hamiltonian," *Nature Physics* 13, no. 12 (2017): 1,195–1,200, doi:10.1038/nphys4243; and Alexei Kitaev, "Fault-Tolerant Quantum Computation by Anyons," *Annals of Physics* 303, no. 1 (2003): 2–30, doi:10.1016/S0003-4916(02)00018-0. For more about this, see the discussion of "Noise Immunity" in the next section below.
35. Siddhardh Morampudi et al., "Statistics of Fractionalized Excitations through Threshold Spectroscopy," *Physical Review Letters* 118, no. 22 (2017), doi:10.1103/physrevlett.118.227201.
36. Martin Greiter and Frank Wilczek, "Heuristic Principle for Quantized Hall States," *Modern Physics Letters B* 4, no. 16 (1990): 1,063–69, doi:10.1142/s0217984990001331.
37. See the excellent article "Quipu" in *Wikipedia*.
38. It is entertaining and enlightening to do a Google search on "braids" and then choose the Images results. You'll quickly appreciate that braids can get very complicated, very fast.
39. Torsten Karzig et al., "Scalable Designs for Quasiparticle-Poisoning-Protected Topological Quantum Computation with Majorana Zero Modes," *Physical Review B* 95, no. 23 (2017), doi:10.1103/physrevb.95.235305.
40. Elaborate braided hairstyles would be hopeless to maintain if they could be undone by passing breezes or nodding of the head.

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