John Horton Conway

The Game of Life

Daniel Kleitman

THE BRILLIANT mathematician John Horton Conway died of complications from COVID-19 in April 2020. He was eighty-two years old. His career included significant contributions to many domains and earned him some of the most prestigious prizes awarded in mathematics.

Most of us experience mathematics as a collection of rules and results, and it is widely thought that a mathematician is someone who has expert knowledge of those rules and results. What he does with that knowledge is a mystery. This view is not entirely wrong, but it is misleading. Mathematicians achieve success through their research, and mathematical research has two aspects. The first involves the creation of problems—often, but not always, conjectures about why two dissimilar mathematical structures have the same size. The second involves proving or disproving such conjectures. Conway was a great master of the second kind. He preferred to ignore both the received wisdom and the received opinion about a mathematical field in order to strike out on his own. He would reinvent the subject from scratch, developing his own ideas about it, and in Conway's hands, this often led to important progress.

As a small child, Conway was fascinated by numbers. He wondered where the integers went, and calculated the values of very many powers of two for his own amusement. These early interests and experiences seem to have caused the parts of his brain that involved mathematical reasoning to develop extensively. His mathematical talents were always extraordinary, and his academic success brought him to the University of Cambridge, where he continued to excel. During his time as a graduate student, Conway solved a problem proposed by his advisor, determining that all integers could be represented as the sums or differences of 37 fifth powers. As it turned out, another researcher published the same result before Conway wrote up his own work.

Cambridge retained Conway as a postdoctoral fellow, and he began to study infinite ordinal numbers, a subject that never really appealed to him. Over time, his personality developed, and a shy introvert gradually became a gregarious extrovert and a spellbinding lecturer. He spent a great deal of time with graduate students, playing games, analyzing game strategies, and inventing games, activities that some considered time wasting.

In 1967, Conway was almost thirty, the age at which mathematicians are said to start losing their powers. He had not yet published a research paper, and he was starting to feel the pangs of depression. Suddenly, his life changed. A mathematician named John Leech visited Cambridge and tried to interest the mathematicians in a problem related to his own work. Most were busy with their own activities; they listened politely then promptly forgot Leech's problem. Conway was seeking an outlet for his talents and vowed to solve it. And he did.

Almost everyone knows that seven pennies can be arranged so that six of them surround the seventh, touching it and their two neighbors. With an infinite number of pennies, the plane can be filled so that each penny touches six of its neighbors.

The analogous arrangement cannot be made in three dimensions, using marbles instead of pennies.

Why not?

In three dimensions, up to twelve marbles of the same size can be placed around a given marble, but they will not all touch their neighbors. Six marbles, for example, can be placed around that given marble so that all seven centers lie in one plane, as in two dimensions. On either side of these six marbles, three more marbles, all touching one another, can be placed, for a total of twelve marbles around that given marble. This is the most that there is room to accommodate. But all of the additional marbles cannot, at the same time, touch two of the first six.

Surprisingly, there exists a set of neighbors—all meeting the first marble, or ball, and also meeting all their neighbors—that can be extended indefinitely, as in one or two dimensions, in eight and twenty-four—spaces described by eight or twenty-four numbers—but in no other dimensions at all. Leech had found a way to describe the locations of the centers of all the balls that met all their neighbors in twenty-four dimensions, now known as the Leech lattice, and he wanted to verify that it worked. He asked if someone at Cambridge could describe the symmetries of his construction.

The construction itself is quite large, with each ball having 196,560 neighbors, all of which it touches. As far as its symmetries go, each of these neighbors can be mapped into each of the others. If two neighbors are fixed, there are on the order of 196,560 ways of arranging the others to produce a symmetry, making the total number of symmetry operations roughly 8×10^{18}, all told.

Conway resolved to find a comprehensible description of these symmetries, which form what mathematicians call a group. To determine a group, it is sufficient to find its generators, members from which all the other members can be produced by means of repetitions and combinations. In a short time, he had an idea that worked, and he was able to find such a description of the group.

At the time, group theory was a hot topic in mathematics. Many mathematicians were working to determine all of the finite simple groups, the very atoms of finite group theory. There were a number of infinite families of such groups that were symmetries of standard mathematical constructs that existed in all dimensions. There were also a few others, known as sporadic groups, and these, it was hoped, were all of them. But nobody could prove that, and for good reason.

The group that Conway described, and others closely related to it, were new simple groups that no one had previously encountered. In a relatively short time, mathematicians closed this story by demonstrating that Conway's groups were the end of it. Old and new simple groups included all possible simple groups.

His result was not a mere curiosity. It provided the missing link that allowed the resolution of the big problem of group theory. That resolution opened a new way to prove things about groups: if something holds for all simple groups, it has to be true for all groups.

Conway quickly rose from being a time waster to being a "magical genius," as his colleague Simon Kochen would later refer to him.[1] Even Conway's time wasting became exciting. During the 1950s, Stanisław Ulam and John von Neumann introduced mathematicians and computer scientists to cellular automata. Consider an infinite grid that contains an initial configuration of 0s and 1s. For each cell, the rule that determines its successor depends on its neighbors. It was thus that Conway introduced his famous Game of Life. The rule governing the Game of Life was very simple: a cell gets a 1 only if either the sum of the entries in the eight surrounding cells is three, or the sum of the entries in the cell and the surrounding cells is three. All sorts of wonderful things happen, depending on the starting configuration. Anyone can set this up on a spreadsheet, and there are websites that can display the development over time. Fiddling with this set of rules has been immensely popular and has made Conway's name well known outside the mathematics community.

The Game of Sprouts is another of his popular time wasters. It is a two-person game in which some number of points are drawn in the plane. Players alternately draw an arc between two points and add a new point somewhere in the middle of the arc. A point can meet at most three lines, two of which may be both ends of the same line.

Another of Conway's concoctions is the concept of surreal numbers. In the nineteenth century, Richard Dedekind defined real numbers in terms of the rational ones by identifying a real number with the set of rational numbers less than it. The real numbers between 0 and 1 can be described as a decimal point followed by all infinite sequences of 0s and 1s, in binary notation. The rational numbers less than any real number include all finite-length prefixes of that real number, and the set of all of these defines the real number.

Conway went a step further than Dedekind. For any two sets of numbers A and B with the largest in A less than the smallest in B, he defined $< A \mid B >$ as follows. When A and B are both the empty set, then $< \mid >$ is zero. If B is the empty set and the largest defined number in A is n, then $< A \mid B >$ is $n + 1$. When the empty set lies to the right of the vertical bar and n is the largest number to the left of it, then the result is $n + 1$. When the largest number in A is x and the smallest in B is y, then $< A \mid B >$ is the number halfway between them. If the process of definition is continued infinitely in all possible ways, one gets all the integers, all the fractions whose denominators are powers of 2, and eventually all the fractions and all the real numbers. So far, this construction is somewhat like Dedekind's. Conway then took a step further to include $< A \mid B >$ when A consists of zero and B consists of all positive rational, or real, numbers. This is something new. It has been called *epsilon*, because it is just like the symbol for an infinitesimal quantity that tortures calculus students. The resulting numbers are termed *surreal*. Rules for adding, multiplying, and otherwise combining surreal numbers are consistent with those we expect, and many things can be proven about them. This was a way to introduce an infinitesimal into number systems in a natural sort of way.

These subjects represent only a small portion of Conway's contributions to mathematics. Each led to collaborations with other mathematicians who saw ways to use his approaches in the problems of interest to them.

Conway left Cambridge and went to Princeton in 1987, where he remained until his death. He had a gift for making complicated mathematics appear simple, both in speaking and in the many popular books he wrote.[2] The breadth and the depth of his contributions to mathematics were extraordinary.

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1. Catherine Zandonella, "Mathematician John Horton Conway, a 'Magical Genius' Known for Inventing the 'Game of Life,' Dies at Age 82," *Princeton.edu*, April 14, 2020.
2. Among his volumes are *The Book of Numbers* (New York: Copernicus, 1996); *On Numbers and Games* (New York: Academic Press, 1976); *Regular Algebra and Finite Machines* (London: Chapman and Hall, 1971); *Sphere Packings, Lattices, and Groups* (New York: Springer Verlag, 1988); as well as several co-authored books.

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