Tōhoku

Rick Jardine

The publication of Alexander Grothendieck's paper, "Sur quelques points d'algèbre homologique" (Some Aspects of Homological Algebra), in the 1957 number of the *Tōhoku Mathematical Journal*, was a turning point in homological algebra, algebraic topology and algebraic geometry.[1] The paper introduced ideas that are now fundamental; its language has withstood the test of time. It is still widely read today for the clarity of its ideas and proofs. Mathematicians refer to it simply as the *Tōhoku* paper.

One word is almost always enough—*Tōhoku*.

Grothendieck's doctoral thesis was, by way of contrast, on functional analysis.[2] The thesis contained important results on the tensor products of topological vector spaces, and introduced mathematicians to the theory of nuclear spaces. Grothendieck published at least twelve other substantial papers on subjects related to his thesis between 1950 and 1957, but by 1955, his attention had already shifted from functional analysis to homological algebra and algebraic geometry. His letters to Jean-Pierre Serre reveal how advanced his thinking had become. There are profound questions throughout about Steenrod operations and Künneth formulae in sheaf cohomology.[3]

Some French analysts, such as Roger Godement, had also been interested in homological algebra and sheaf theory. But Grothendieck left analysis altogether, and so executed a radical intellectual pivot. It was a pivot comparable to that executed by Franz Kline, as his representational work gave way to abstract expressionism.

Grothendieck was always an axiomatic thinker, and early on he adopted a categorical point of view. First introduced by Samuel Eilenberg and Saunders Mac Lane in the late 1940s, a category is a collection of similar mathematical objects, with arrows in flight, or morphisms, between them.[4] There are many, many categories. Groups are an example; and group homomorphisms, their arrows in flight. The mathematical universe is now organized in these terms.

Once introduced, categorical language, like Esperanto, proved easy to learn. Mathematicians quickly acquired fluency; the topologists first, then all the rest. Language learning led to great advances: the axiomatic description of homology theory, the theory of adjoint functors, and, of course, the concepts introduced in *Tōhoku*.[5]

This great paper has elicited much by way of commentary, but Grothendieck's motivations in writing it remain obscure. In a letter to Serre, he wrote that he was making a systematic review of his thoughts on homological algebra.[6] He did not say why, but the context suggests that he was thinking about sheaf cohomology. He may have been thinking as he did, because he could. This is how many research projects in mathematics begin. The radical change in Grothendieck's interests was best explained by Colin McLarty, who suggested that in 1953 or so, Serre inveigled Grothendieck into working on the Weil conjectures.[7] The Weil conjectures were certainly well known within the Paris mathematical community, and they motivated much of the research done by geometers and number theorists.

André Weil published his conjectures in 1949.[8] They are concerned with the relationship between local zeta functions and points on algebraic varieties over finite fields. The varieties hovering over the finite fields contain a finite number of rational points; and they contain, as well, points from every field subsuming the finite field. Coefficients of any local zeta function are derived from the number of these points. Weil conjectured first that the zeta functions were rational; second, that they satisfied a functional equation; and third, that their zeros fell along the fabled critical line that plays so conspicuous a role in the Riemann hypothesis.

Weil's fourth conjecture expresses a relationship between the Betti numbers of a projective algebraic variety, one defined over a subfield of the complex numbers, and its rational points. There is a grand audacity in this: the Betti numbers are ranks of rational homology groups, and the conjecture compares them to the solutions of a system of polynomials in finite fields. Weil suggested that a link must exist between topological and arithmetic invariants. This was an intoxicating idea in the early 1950s, when there was no machinery for creating such links.

It is intoxicating still.

The methods used for proving the Weil conjectures are fundamental for the mathematics that functions at the confluence of geometry, number theory, and topology. Much has changed since the appearance of *Tōhoku*. We understand the landscape better. Still, the relations between finite fields and number fields continue to generate fascinating theoretical and computational problems. Cohomological and homotopic invariants arising from number fields are usually difficult to compute, and it has become a standard method to approximate these invariants with calculations over finite fields.

Weil quite understood that a proof of his conjectures might turn on the definition of a suitable cohomology theory. Mathematicians thought that the Weil cohomology might be a kind of sheaf cohomology, one satisfying a Lefschetz fixed-point formula, among other properties. Not so; not quite. Sheaf cohomology, as it was understood in the early 1950s, was an artifact of algebraic topology, and so needed an underlying topological space. The underlying space could not be standard because it had to encode *both* topological and arithmetical information. What was needed, although it was not known at the time, would eventually be called the étale site of a scheme. It emerged by means of two major intellectual advances: the generalization of an algebraic variety into a scheme, and the promotion of a generalized space into a topos.

These ideas were worked out by Grothendieck and his collaborators in less than ten years. Pierre Deligne demonstrated the fourth, and most difficult, of the Weil conjectures in 1974.[9]

To be sure, schemes were in the air. They are among the mathematical ideas, Serre seems to have said, that were never invented, but always acknowledged.[10] Grothendieck gave schemes their first explicit exposition in a series of volumes under the title *Éléments de géométrie algébrique* (Elements of Algebraic Geometry, or ÉGA).[11] Geometric topos theory first appeared in Michael Artin's Harvard notes of 1962, and the basics of the étale site and étale cohomology were worked out in great detail in the year that followed.[12] The four volumes of the *Séminaire de Géométrie Algébrique du Bois Marie* (Seminars in Algebraic Geometry, or SGA4)—in which these ideas were introduced, expressed, polished, and advanced—were published a decade later.[13]

Sheaf cohomology theory was under active development in the early 1950s. Eilenberg and Mac Lane had a theory of group cohomology, but sheaf cohomology was not yet defined in terms of derived functors. It suffered in some key respects. In plain fact, sheaf cohomology had not yet been properly categorified, to use a word that is as widely deplored as it is widely used.

Tōhoku solved the initial problems of sheaf cohomology. The development of this new theory was the first decisive step towards proving the Weil conjectures. Grothendieck's review of his thoughts on homological algebra, mentioned in his letter to Serre, ultimately became *Tōhoku*. He had the habits of an autodidact; it was part of his working style to reinvent theories from scratch. Finding himself in Kansas in 1955 and 1956—of all places!—he lacked access to Henri Cartan and Eilenberg's treatise on homological algebra. He did not know, or he ignored, David Buchsbaum's "Exact Categories and Duality."[14] Communication among mathematicians was difficult.

Grothendieck advanced on his own.

Tōhoku set abelian sheaf theory within an abstractly-defined abelian category. An abelian category is an additive category. Its morphisms have the structure of an abelian group. Two axioms are in charge. The first says that abelian categories have infinite direct sums; the second, that infinite direct sums are compatible with fixed object intersections.

Abelian sheaf categories satisfy all of these conditions, because they are satisfied by the category of abelian groups; the associated sheaf functor is exact. The required set of generators is given by representable functors. Sheaf cohomology groups are then defined as derived functors of the global sections functor.

The paper's chief result, which I shall call *Grothendieck's theorem*, is that, under these conditions, every object of the category has an injective resolution, which can be used to define derived functors in the sense of good old-fashioned homological algebra.

You just saw a proof go by—welcome to topos theory.

"The associated sheaf functor is exact." Note the use of the word *exact*. It has a precise sense, one introduced in *Tōhoku* and destined to become a central feature of both homological algebra and topos theory. This concept eventually led to Jean Giraud's famous theorem that a topos (or a sheaf category) can be characterized by its exactness properties. Giraud's theorem seems abstract, but its proof gives a very useful and explicit recipe for constructing a category of sheaves from a category that only looks like a category of sheaves. This observation can now be viewed as a basis for both equivariant homotopy theory and Galois cohomology theory.

Within the structure given by the definitions, the proof of Grothendieck's theorem is easy, and has become ubiquitous despite the fact that it depends on transfinite induction. It is a special case of the small object argument, which Daniel Quillen would later introduce in his axiomatization of homotopy theory, and that is now part of the basic tool kit of algebraic topology.[15]

It is interesting to observe that Grothendieck used his own very sensible and now standard definition of a category, in which morphisms between two fixed objects form a set, rather than something bigger. This was not Eilenberg and Mac Lane's way. At the time that *Tōhoku* appeared, Mac Lane was still saying that categories of this sort were locally small. If these categories were locally small, then plainly others were not.

He came to adopt Grothendieck's formulation only later.[16]

There has been a longstanding debate about whether mathematics rests on categories or on sets.[17] Grothendieck's approach to category theory was firmly grounded in naive set theory; he made constant use of sets of morphisms, sets of generators, set-indexed limits and colimits, transfinite induction, and Zorn's lemma. There is no evidence that any of this made him anxious. It is a practical approach, very much in use today among geometers and topologists; and Grothendieck's definitions and constructions do not easily admit interpretation outside of set theory.

Grothendieck later proposed admitting universes among the sets. The universes go beyond the usual rules of set theory.[18] At first blush, this concept seems attractive, but in the end there is no practical reason to use it. Aside from the unpalatable fact that one is tacking an extra assumption onto set theory, universes tend to conceal cardinal arithmetic and so often lead to mistakes. One is generally obliged to ensure that constructions involving the small stuff stay small in a quantifiable way.

The foundational discussion is now still more complicated. Abstract homotopy theory has recently emerged as a rival for category theory as a foundation for mathematics. Perhaps this is not so surprising, since category theory and abstract homotopy theory have become the same field.

I recall Mac Lane referring to "my old enemy Grothendieck" over coffee one morning. I was too young at the time to ask him what he meant, and then the conversation progressed to a joint grumble about the complexity of the first two volumes of SGA4. Mac Lane and I shared the conviction that easy mathematics, and particularly category theory, should be expressed simply.

Grothendieck and Mac Lane were opinionated men.

The *Tōhoku* contains a general description of spectral sequences in abelian categories, and the particular case of a spectral sequence for the composite of two exact functors that is now called the Grothendieck spectral sequence.

The theory of spectral sequences has become one of the primary tools of modern mathematics. A spectral sequence is an exact sequence machine, one used to compute homological invariants in stages. The device can be initially daunting for graduate students, but it is easily mastered.

Both spectral sequences and sheaf theory were introduced by Jean Leray while he was a prisoner of war in Austria. Announcements of his work were published in 1946. The spectral sequence machinery was still relatively new when *Tōhoku* was written.[19] Grothendieck recreated the theory within the context of an abelian category, in a form that is still quite readable today. The Leray spectral sequence for the push-forward of a sheaf—one clunker of a term—which was the original example, is a special case of the Grothendieck spectral sequence. The functors to be composed in this case are the push-forward and global sections.

The fundamental abstract theory of the *Tōhoku* is presented in its first two chapters. The remaining three chapters are devoted to the special case of sheaves and sheaf cohomology theory for topological spaces.

In the third chapter of his paper, Grothendieck observed that there is an easy enough proof that the category of abelian sheaves on a topological space has sufficiently many injectives. The classical Godement treatment was constructed from stalks. The real strength of Grothendieck's theorem would only appear later. Generalized sheaf categories might not have a theory of stalks. Many sheaf categories, including the étale topos, do have such stalks, but there are important examples that do not. A prime example is the flat topos, currently in use for much of stack theory (stacks without stalks), and for the theory of topological modular forms in stable homotopy theory.

The definition of a sheaf as a presheaf satisfying patching conditions made its first published appearance in the *Tōhoku*. In modern terms, Grothendieck showed that this definition is equivalent to the definition of a sheaf as a space fibered over the base space (*éspace étalé*), with a topology defined by local homeomorphisms.

One good turn deserves another. In showing that the two definitions coincided, Grothendieck introduced the much larger concept of categorical equivalence. The elements of the definition and the proof of equivalence were already in place, explicitly in Grothendieck's unpublished 1955 National Science Foundation report, and tangentially in Serre's 1955 paper, "Faisceaux algébriques cohérents," (Coherent Algebraic Sheaves).[20] The modern definition of a sheaf also appears in Godement's book, which was based on notes for a series of lectures given at the University of Illinois between 1954 and 55.[21]

In the third chapter of the *Tōhoku*, Grothendieck compares his derived functor definition of sheaf cohomology to the existing definitions of sheaf cohomology for spaces. He proves an acyclicity criterion for flabby (*flasque*) and soft (*mou*) sheaves, objects that have no higher derived functor cohomology groups. It follows, via a spectral sequence argument, that the cohomology groups defined by flabby resolutions coincide up to natural isomorphism with derived functor cohomology groups.

With this result, Grothendieck's new construction of abelian sheaf cohomology theory was firmly in place.

Grothendieck also compared sheaf cohomology with Čech cohomology. It is well known that Čech cohomology and sheaf cohomology coincide for locally acyclic spaces. It has arbitrarily small contractible neighborhoods, as in the calculus. Grothendieck proved that local acyclicity implies the coincidence of Čech and sheaf cohomology, by using the first instance of a descent spectral sequence for sheaf cohomology.

Descent is now a very big deal in modern mathematics. Grothendieck's descent spectral sequence serves to compute sheaf cohomology from cohomology on patches over an open cover and all of its finite intersections. The Čech resolution is homotopically equivalent to the space it is covering; the injective resolution of an abelian sheaf satisfies descent. There is no light between the Čech resolution and its space.

This formulation has evolved in the past thirty years, in the context of a non-abelian homotopy theory for combinatorial structures in sheaves, called simplicial sheaves (or presheaves). These are among Grothendieck's late-in-life thoughts. Homotopy theory for simplicial sheaves first appeared in a letter from André Joyal to Grothendieck in 1984. The key steps along the way included Luc Illusie's formulation of a quasi-isomorphism of simplicial sheaves, and Barr's theorem, which creates fat points for a topos where there is no theory of stalks.[22]

Sheaf cohomology theory has evolved from the *Tōhoku* to a special case of the homotopy theory of these constructs, mirroring the situation in classical algebraic topology. Their cohomology groups are represented in the homotopy category by certain types of spaces, called Eilenberg-Mac Lane spaces.

From this point of view, sheaf cohomology becomes a special case of generalized cohomology theories represented by simplicial sheaves, or even presheaves of spectra. Algebraic K-theory of spectra was the primary, motivating example. The Lichtenbaum–Quillen conjecture affirmed that algebraic K-groups can be recovered, most of the time, from étale cohomology groups via a descent spectral sequence. The theory satisfies a descent condition. In the best of all possible worlds, it behaves like an injective resolution of an abelian sheaf.

Algebraic K-theory was introduced by Grothendieck in his proof of the Riemann–Roch theorem, while higher algebraic K-theory was defined by Quillen.[23] Algebraic K-theory presheaf of spectra followed much later, in a series of steps.

The Lichtenbaum–Quillen conjecture has been proved: it follows from the Bloch–Kato conjecture, which is a statement about the Galois cohomology of fields. Vladimir Voevodsky received the Fields Medal in 2002 for his work on motives, motivic cohomology, and the Bloch–Kato conjecture. His proof involved a fundamentally new type of stable homotopy theory, which is often called motivic stable homotopy theory. Voevodsky's proof also strongly depends on a sharpening of some of Grothendieck's ideas from the 1960s.

The étale cohomological descent approach to the Lichtenbaum–Quillen conjecture was not the only descent story in algebraic K-theory. For smooth schemes, K-theory is recoverable by descent from the Zariski topology and the Nisnevich topology.

These are major results.

The Baum–Connes conjecture in noncommutative geometry can be viewed as a descent statement, and so, too, can the theory of topological modular forms in stable homotopy theory.

We now understand the theory of stacks and non-abelian cohomology theory in a much more tractable, homotopical way. Stack theory is, I suggest, the modern geometric theory of symmetries. The objects themselves were a bit mystical for quite a while: one of the classical definitions says that a stack is a sheaf of groupoids that satisfies an effective descent condition: groupoids in sections are determined by patched-together groupoids. In this way, groupoids become multi-object groups, or categories in which all morphisms have inverses. Effective descent is equivalent to homotopy theoretic descent, and stacks are homotopy types of sheaves or presheaves of groupoids. Furthermore, all elements in non-abelian cohomology can be explicitly constructed from suitable defined cocycles. The new theories of n-stacks are variations on this theme.[24] Stacks and higher stacks appear as objects of study in multiple disciplines, including algebraic geometry, algebraic topology, number theory, analytic geometry, and theoretical physics.

Grothendieck's approach to the homological algebra of abelian sheaves has evolved in multiple ways. The construction of Čech cohomology would later be revised to include all hypercovers. Looser definitions have led to the Verdier hypercovering theorem, which appeared in SGA4 and says that sheaf cohomology can always be recovered from this expanded version of Čech theory.

Artin and Barry Mazur used this idea in their construction of étale homotopy theory, introducing homotopic theoretic techniques into étale topology.[25] Étale homotopy theory was the basis of the original descriptions of the Lichtenbaum–Quillen conjectures of Eric Friedlander and William Dwyer, and it still gives some of the best information about the absolute Galois group of a scheme.[26] The Verdier hypercovering theorem, on the other hand, was more general, and was the genesis of the local fibration approach to the homotopy theory of simplicial sheaves, which started to appear in Ken Brown's thesis.[27]

A hypercover is a type of resolution, and the Verdier hypercovering theorem was, for many years, the projective resolution side of sheaf cohomology theory. The model structure given by Quillen for chain complexes was an early, convincing representation of his axioms, as it allowed one to simply rewrite the basic homological algebra of chain complexes in homotopical terms.[28] Sheaves of chain complexes are another matter. The *Tōhoku* notes that categories of abelian sheaves do not, in general, have a theory of projective resolutions. As a result, one cannot copy Quillen's definition of a fibration for ordinary chain complexes to chain complexes of sheaves. It took a while to find a workaround. Let Joyal's model structure for simplicial sheaves induce a model structure for simplicial abelian sheaves and hence for sheaves of chain complexes. That is the modern workaround. In this setting, projective resolutions are replaced by cofibrant resolutions. The abelianizations of hypercovers are an example.

We end with a purely homotopical approach to the construction of the derived category of abelian sheaves, one that is consistent with the homotopical theory of simplicial sheaves and presheaves in the same way as the derived category of chain complexes is consistent with classical homotopy theory.

This can be made much more complicated; mathematicians are eager to try.

Grothendieck's *Tōhoku* paper was the start of a long period of axiomatic machine building which, in the presence of highly non-trivial inputs from algebraic geometry, ultimately led to the proof of the Weil conjectures, and later to the proofs of the Bloch–Kato and the Lichtenbaum–Quillen conjectures. The axiomatic approach has come to characterize much of modern mathematics. The methods and results which evolved from *Tōhoku* have fused ideas and techniques from homotopy theory, geometry, and number theory.

The machines themselves are context independent combinatorial constructions. They go anywhere. And they are applicable in multiple parts of the mathematical sciences, including traditional mathematical areas, but also in other disciplines. Such applications have started to appear in theoretical physics, topological data analysis, models for parallel processing systems, computer language design, and studies of sensor networks.

The ideas and methods of the *Tōhoku* opened a particularly rich and still expanding vein of mathematical and scientific inquiry.

Unending significance.

Rick Jardine is a Professor of Mathematics at the University of Western Ontario in London, Canada.

1. Alexander Grothendieck, "Sur quelques points d'algèbre homologique" (Some Aspects of Homological Algebra), *Tōhoku Mathematical Journal* 9, no. 3 (1957): 119–221.
2. Alexander Grothendieck, "Produits tensoriels topologiques et espaces nucléaires," *Memoirs of the American Mathematical Society* 16 (1955).
3. Pierre Colmez and Jean-Pierre Serre, *Correspondance Grothendieck-Serre* (Paris: Société Mathématique de France, 2001). A bilingual edition is *Grothendieck-Serre Correspondence*, trans. Catriona Maclean (Providence, RI: American Mathematical Society, 2004).
4. Samuel Eilenberg and Saunders Mac Lane, "General Theory of Natural Equivalences," *Transactions of the American Mathematical Society* 58 (1945): 231–94.
5. Samuel Eilenberg and Norman Steenrod, *Foundations of Algebraic Topology* (Princeton, NJ: Princeton University Press, 1952); Daniel Kan, "Adjoint Functors," *Transactions of the American Mathematical Society* 87 (1958): 294–329.
6. Pierre Colmez and Jean-Pierre Serre, *Grothendieck-Serre Correspondence*, trans. Catriona Maclean (Providence, RI: American Mathematical Society, 2004), 6.
7. Colin McLarty, "The Rising Sea: Grothendieck on Simplicity and Generality," in Jeremy Gray and Karen Parshall, eds., *Episodes in the History of Modern Algebra (1800–1950)* (Providence, RI: American Mathematical Society, 2007), 301–25.
8. André Weil, "Numbers of Solutions of Equations in Finite Fields," *Bulletin of the American Mathematical Society* 55 (1949): 497–508.
9. See Pierre Deligne, "La conjecture de Weil. I." *Institut des Hautes Études Scientifiques Publications Mathématiques* 43 (1974): 273–307; Pierre Deligne, "La conjecture de Weil. II." *Institut des Hautes Études Scientifiques Publications Mathématiques* 52 (1980): 137–252.
10. Colin McLarty, "The Rising Sea: Grothendieck on Simplicity and Generality," in Jeremy Gray and Karen Parshall, eds., *Episodes in the History of Modern Algebra (1800–1950)* (Providence, RI: American Mathematical Society, 2007), 301–25.
11. Alexander Grothendieck, "Éléments de géométrie algébrique. I. Le langue des schémas." *Institut des Hautes Études Scientifiques Publications Mathématiques* 4 (1960): 1–228.
12. Michael Artin, *Grothendieck Topologies: Notes on a Seminar* (Cambridge, MA: Harvard University, Department of Mathematics, 1962).
13. See Alexander Grothendieck, ed. "Théorie des topos et cohomologie étale des schémas (SGA4). Tome I: Théorie des topos." *Lecture Notes in Mathematics* (Springer-Verlag) 269 (1972); Alexander Grothendieck, ed. "Théorie de topos et cohomologie étale des schémas (SGA4). Tome 2." *Lecture Notes in Mathematics* (Springer-Verlag) 270 (1972); Alexander Grothendieck, ed. "Théorie des topos et cohomologie étale des schémas (SGA4). Tome 3." *Lecture Notes in Mathematics* (Springer-Verlag) 305 (1973).
14. Henri Cartan and Samuel Eilenberg, *Homological Algebra* (Princeton, NJ: Princeton University Press, 1956); David Buchsbaum, "Exact Categories and Duality," *Transactions of the American Mathematical Society* 80 (1955): 1–34.
15. Daniel Quillen, *Lecture Notes in Mathematics: Homotopical Algebra* (Berlin, Heidelberg: Springer-Verlag, 1967).
16. Saunders Mac Lane, "Categorical Algebra," *Bulletin of the American Mathematical Society* 71 (1965): 40–106; Saunders Mac Lane, "Locally Small Categories and the Foundations of

Set Theory," in *Infinitistic Methods (Proceedings of Symposia on Foundations of Mathematics, Warsaw, 2-9 September 1959)* (Oxford, UK: Pergamon, 1961), 25–43.

17. Solomon Feferman, "Categorial Foundations and Foundations of Category Theory," in Robert Butts and Jaakko Hintikka, eds., *Logic, Foundations of Mathematics, and Computability Theory* (Dordecht, Holland: D. Reidel, 1977), 149–69.
18. Pierre Gabriel, "Des catégories abéliennes," *Bulletin de la Société Mathématique de France* 90 (1962): 323–448.
19. Jean Leray, "L'anneau d'homologie d'une représentation," *Comptes rendus de l'Académie des Sciences Paris* 222 (1946): 1,366–68; Jean Leray, "Stucture de l'anneau d'homologie d'une représentation," *Comptes rendus de l'Académie des Sciences Paris* 222 (1946): 1,419–22.
20. Alexander Grothendieck, *A General Theory of Fibre Spaces with Structure Sheaf*, 2nd ed., National Science Foundation research project on geometry of function spaces, Department of Mathematics, University of Kansas, 1958; Jean-Pierre Serre, "Faisceaux algébriques cohérents," *Annals of Mathematics* 61, no. 2 (1955): 197–278.
21. Roger Godement, *Topologie algébrique et théorie des faisceaux* (Paris: Hermann, 1958); John Gray, "Fragments of the History of Sheaf Theory," in *Applications of Sheaves (Proc. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., University of Durham, 1977)*. (Berlin: Springer-Verlag, 1979), 1–79; Ralph Krömer, *Tool and Object: A History and Philosophy of Category Theory* (Basel: Birkhäuser Verlag, 2007).
22. Luc Illusie, *Lecture Notes in Mathematics: Complexe cotangent et déformations, I.* (Berlin, Heidelberg: Springer-Verlag, 1971); Michael Barr, "Toposes Without Points," *Journal of Pure and Applied Algebra* 5, no. 3 (1974): 265–80.
23. Armand Borel and Jean-Pierre Serre, "Le théorème de Riemann–Roch," *Bulletin de la Société Mathématique de France* 86 (1958): 97–136; Daniel Quillen, *Lecture Notes in Mathematics: Higher Algebraic K-Theory: I.* (Berlin, Heidelberg: Springer-Verlag, 1973), 85–147.
24. Carlos Simpson, "Descent," in Leila Schneps, ed., *Alexandre Grothendieck: A Mathematical Portrait* (Somerville, MA: International Press, 2014), 83–141.
25. Michael Artin and Barry Mazur, *Lecture Notes in Mathematics: Etale Homotopy* (Berlin, Heidelberg: Springer–Verlag, 1969), 100; Eric Friedlander, *Étale Homotopy Theory of Simplicial Schemes. (AM-104)* (Princeton, NJ: Princeton University Press, 1982), 104.
26. Eric Friedlander, "Étale *K*-theory. II. Connections with algebraic *K*-theory" *Annales scientifiques de l'École Normale Supérieure (4)* 15, no. 2 (1982): 231–256; William Dwyer and Eric Friedlander, "Algebraic and Étale *K*-theory," *Transactions of the American Mathematical Society* 292, no. 1 (1985): 247–80.
27. Kenneth Brown, "Abstract Homotopy Theory and Generalized Sheaf Cohomology," *Transactions of the American Mathematical Society* 186 (1974): 419–58.
28. Daniel Quillen, *Lecture Notes in Mathematics: Homotopical Algebra* (Berlin, Heidelberg: Springer–Verlag, 1967).

Published on July 13, 2015

https://inference-review.com/article/tohoku